A parametric equation of a line had a form

$$\langle x, y, z \rangle = \langle a+v_1 t,\ b+v_2 t,\ c+v_3 t \rangle.$$

We can think of the right hand side as a vector function, namely something that gives a vector when you plug a number into t.

$$\text{number } t \longmapsto \text{vector } \vec{r}(t)$$

This gives rise to more general curves.

The components of the vector function are themselves functions and are called component functions.

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$
$$= f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}.$$

Many properties of the ordinary functions have their analogues for the vector functions.

- The vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is defined at $t=a$ if $f(t), g(t), h(t)$ are all defined at $t=a$.
- The domain of the vector function $\vec{r}(t)$ is the collection of all values of t at which $\vec{r}(t)$ is defined.

- For $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, if all

$\lim_{t\to a} f(t)$, $\lim_{t\to a} g(t)$, $\lim_{t\to a} h(t)$ exist, then

$$\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle.$$

- We say $\vec{r}(t)$ is continuous at $t=a$ if $\vec{r}(t)$ is defined at $t=a$ and $\vec{r}(a) = \lim_{t\to a} \vec{r}(t)$

<u>Example</u> The domain of

$$\vec{r}(t) = \langle \sqrt{t}, \ln(5-t), t^2 \rangle$$

is such that all three expressions makes sense.

$$\left.\begin{aligned} \sqrt{t} &: t \geq 0 \\ \ln(5-t) &: 5-t > 0 \\ t^2 &: \text{Any } t \end{aligned}\right] \rightsquigarrow \text{The domain is } \{0 \leq t < 5\}$$

<u>Example</u> The domain of the vector function

$$\vec{r}(t) = \langle \sqrt{t}, \sqrt{-t}, t \rangle$$

is such that

$t \geq 0$, $-t \geq 0$, anything

all three should be satisfied, so $\{t=0\}$ is the domain.

A domain is **bounded** if it doesn't go to infinity.

Examples $\{10 \le t < 20\}$, $\{-2 < t < 1\}$ ...

Non-examples $\{t \ge 0\}$ (t can go to $+\infty$)

A domain is **closed** if it is defined only using $\le, \ge, =$, without using $<, >$, or $\ne$.

Examples $\{10 \le t\}$, $\{2 \le t$ or $t \le -5\}$, $\{t = 0\}$, ...

Non-examples $\{t \ne 0\}$, $\{t > 0\}$, $\{2 > t \ge -3\}$, ...

A domain is **compact** if it is both bounded and closed.

Examples $\{t = 0\}$, $\{2 \le t \le 5\}$, ...

Non-examples $\{2 < t < 5\}$ (not closed)

$\{t \ge 0\}$ (not bounded)

$\{2 \le t \le 5, t \ne 3\}$ (not closed)

The notion of compactness will be important later in the course.

Example $\vec{r}(t) = \left\langle \frac{1}{t}, \frac{1}{t-1}, \frac{1}{t-2} \right\rangle$ is continuous at all t except $t=0$, $t=1$, $t=2$.

Example For $\vec{r}(t) = \left\langle \frac{1}{t}, \frac{t^2-1}{2t^2+1}, \frac{t-1}{t^2+1} \right\rangle$,

$$\lim_{t \to \infty} \vec{r}(t) = \left\langle 0, \frac{1}{2}, 0 \right\rangle.$$

Derivatives of vector functions

Definition For a vector function $\vec{r}(t)$,

$$\boxed{\vec{r}\,'(t) = \frac{d\vec{r}}{dt} = \lim_{s\to 0} \frac{\vec{r}(t+s) - \vec{r}(t)}{s}.}$$

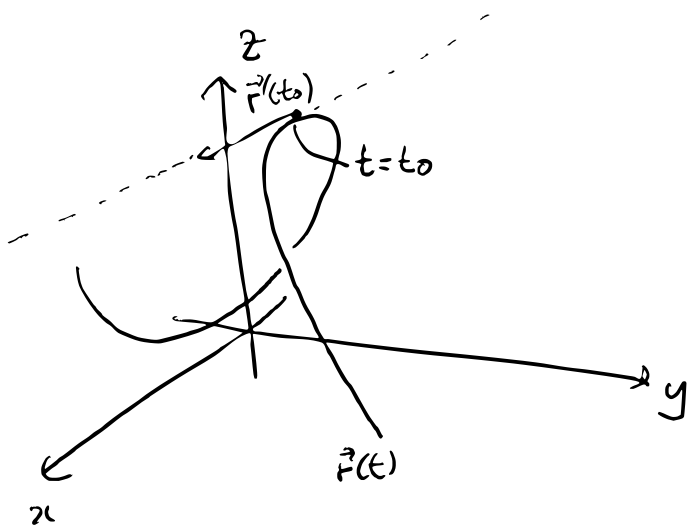
two ways to express the same thing

In particular, $\vec{r}\,'(t)$ is again a vector function:

If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$,

$$\boxed{\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t)\rangle}.$$

Geometrically, $\vec{r}\,'(t)$ indicates the direction of the tangent line.



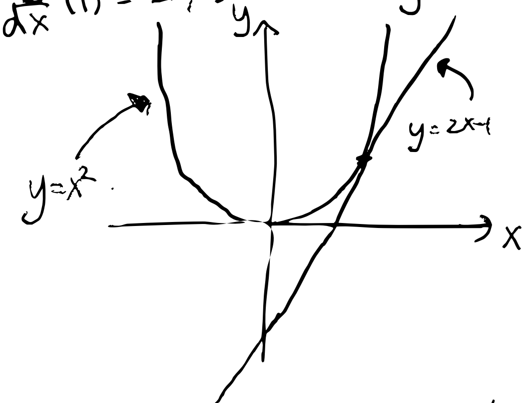
The vector equation for the tangent line at $t = t_0$ is

$$\boxed{\langle x, y, z\rangle = \vec{r}(t_0) + s\vec{r}\,'(t_0)}$$

(s is the time variable).

<u>Example</u> Let's compare this with what we already know. We know that the tangent line of the graph $y = x^2$ at $(1,1)$ has slope $\frac{dy}{dx}(1) = 2$, so the tangent line has the equation

$$\boxed{y = 2x - 1}$$



The same 2D curve can be expressed as a parametric curve

$$\langle x, y\rangle = \langle t^3, t^6\rangle = \vec{r}(t)$$

The point $(1,1)$ correspond to $t = 1$, so the tangent line is the line passing thru $(1,1)$ with the direction

$$\vec{r}\,'(t) = \langle 3t^2, 6t^5\rangle\big|_{t=1} = \langle 3, 6\rangle.$$

So the tangent line can be given by

$$\boxed{\langle x, y\rangle = \langle 3s+1, 6s+1\rangle.}$$

We seem to have obtained two different answers, but in fact they are the same, because

$6s+1 = y = 2x-1 = 2(3s+1)-1 = 6s+1.$ (Reparametrized).

⚠ The vector equation of the tangent line is valid only if $\vec{r}'(t_0) \neq \vec{0}$.

In general, given a curve, different parametrizations travel at different speeds. Therefore, $\vec{r}'(t_0)$ depends on the parametric/vector equation, not on the curve itself.

What is more inherent to the geometric object of the curve is the <u>unit tangent vector</u>.

$$\boxed{\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}}$$

Just as in Calculus, vector functions satisfy similar differentiation rules: for vector functions $\vec{u}(t), \vec{v}(t)$ and a (scalar) function $f(t)$,

- **Sum Rule**: $\dfrac{d}{dt}(\vec{u}(t)+\vec{v}(t)) = \vec{u}'(t)+\vec{v}'(t)$
- **Product Rule I**: $\dfrac{d}{dt}(f(t)\vec{v}(t)) = f'(t)\vec{v}(t)+f(t)\vec{v}'(t)$
- **Product Rule II**: $\dfrac{d}{dt}(\vec{u}(t)\cdot\vec{v}(t)) = \vec{u}'(t)\cdot\vec{v}(t)+\vec{u}(t)\cdot\vec{v}'(t)$
- **Product Rule III**: $\dfrac{d}{dt}(\vec{u}(t)\times\vec{v}(t)) = \vec{u}'(t)\times\vec{v}(t)+\vec{u}(t)\times\vec{v}'(t)$
- **Chain Rule**: $\dfrac{d}{dt}(\vec{u}(f(t))) = f'(t)\,\vec{u}'(f(t))$.

<u>Example</u> (Exercise 13.2.49, 50 of Stewart)

$\vec{u}(t) = \langle \sin t, \cos t, t\rangle$

$\vec{v}(t) = \langle t, \cos t, \sin t\rangle$

Then find $\dfrac{d}{dt}(\vec{u}(t)\cdot\vec{v}(t))$ and $\dfrac{d}{dt}(\vec{u}(t)\times\vec{v}(t))$.

<u>Solutions</u> Note

$$\begin{aligned}\frac{d}{dt}(\vec{u}(t)\cdot\vec{v}(t)) &= \vec{u}'(t)\cdot\vec{v}(t)+\vec{u}(t)\cdot\vec{v}'(t)\\ &= \langle \cos t, -\sin t, 1\rangle\cdot\langle t, \cos t, \sin t\rangle\\ &\quad+\langle \sin t, \cos t, t\rangle\cdot\langle 1, -\sin t, \cos t\rangle\\ &= t\cos t-\sin t\cos t+\sin t\\ &\quad+\sin t-\sin t\cos t+t\cos t = 2(t\cos t-\sin t\cos t\\ &\quad+\sin t).\end{aligned}$$

And

$$\frac{d}{dt}(\vec{u}(t)\times\vec{v}(t)) = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$$

$$= \langle \cos t, -\sin t, 1\rangle \times \langle t, \cos t, \sin t\rangle$$

$$+ \langle \sin t, \cos t, t\rangle \times \langle 1, -\sin t, \cos t\rangle$$

$$= \langle -\sin^2 t - \cos t, -\sin t\cos t + t, \cos^2 t + t\sin t\rangle$$

$$+ \langle \cos^2 t + t\sin t, -\sin t\cos t + t, -\sin^2 t - \cos t\rangle$$

$$= \langle -\sin^2 t - \cos t + \cos^2 t + t\sin t, -2\sin t\cos t + 2t,$$

$$\cos^2 t + t\sin t - \sin^2 t - \cos t\rangle.$$

Integrals of vector functions

Again, one can think of the integral of a vector function

$$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle.$$

The unspecified constant is now a vector!

Indefinite integral:

$$\int \vec{r}(t)\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle + \vec{C}$$

Definite integral:

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$$

Fundamental theorem of calculus

If $\vec{r}(t)$ is a vector function such that

$\vec{r}(t) = \vec{R}'(t)$, then

$$\boxed{\int_a^b \vec{r}(t)\,dt = \vec{R}(b) - \vec{R}(a)}$$